

ON REGULARITY OF THE DISCRETE HARDY-LITTLEWOOD MAXIMAL FUNCTION

FARUK TEMUR

1. INTRODUCTION

In [4] J. Kinnunen proved the boundedness of the Hardy-Littlewood maximal operator given by

$$Mf(x) := \sup_{r>0} \int_{B(x,r)} |f(y)| dy$$

on the Sobolev space $W^{1,p}(\mathbb{R}^n)$ for $1 < p \leq \infty$. Since Mf is never integrable for non-trivial functions this cannot be extended to $p = 1$. However one can ask whether the operator $f \mapsto \nabla Mf$ is bounded from $W^{1,p}(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$. This question, asked by Hajlasz and Onninen in [3], was answered positively for $n = 1$ in the easier case of non-centered maximal function by Tanaka, and for the centered case recently by Kurka; see [6, 4]. Indeed the result of Tanaka was strengthened by J.M. Aldaz and J. Pérez Lázaro in [1] to show

$$(1) \quad \text{var } \widetilde{M}f \leq \text{var } f$$

where $\widetilde{M}f$ is the non-centered maximal function, whereas Kurka derived his answer to the question from the analogous result for the centered one:

$$(2) \quad \text{var } Mf \leq C \text{var } f$$

Consider the discrete Hardy-Littlewood maximal function

$$Mf(n) = \sup_{r \in \mathbb{Z}^+} \frac{1}{2r+1} \sum_{k=-r}^r |f(n+k)|$$

where $f : \mathbb{Z} \mapsto \mathbb{R}$ and \mathbb{Z}^+ denotes non-negative integers. One can similarly define the non-centered version:

$$\widetilde{M}f(n) = \sup_{r,s \in \mathbb{Z}^+} \frac{1}{r+s+1} \sum_{k=-r}^s |f(n+k)|.$$

Although the result of Kinnunen does not meaningfully extend to this setting, the analogue of (1) was showed by Bober, Carneiro, Hughes and Pierce in [2]. In this

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paper we will extend (2) to discrete setting. More precisely define

$$\text{var } f := \sum_{k \in \mathbb{Z}} |f(k+1) - f(k)|$$

for a function $f : \mathbb{Z} \mapsto \mathbb{R}$. Then we prove the following.

Theorem 1. *Let $f : \mathbb{Z} \mapsto \mathbb{R}$ be a function of bounded variation. Then*

$$\text{var } Mf \leq C \text{var } f.$$

It is conjectured in [2] that $C = 1$, but as in Kurka's work we are not able to obtain this constant.

We will adapt ideas developed by Kurka in [5] to discrete setting to obtain our result. The rest of the paper is organized as follows. In the next section we will give definitions necessary for classifying local extrema, and state a lemma that handles the variation arising from one class of local extrema at a time. Using these we will explain main ideas underlying the proof and then prove this lemma. In last three sections the issue of putting all classes together will be dealt with.

2. PRELIMINARIES

Before going to our definitions we first note that it suffices to prove our theorem for non-negative functions. So let $f \geq 0$ be a function defined on integers with bounded variation.

Definition 1. I. *A peak is a system of three integers $p < r < q$ satisfying $Mf(p) < Mf(r)$ and $Mf(q) < Mf(r)$.*

II. *We define the variation of a peak $\mathbb{p} = \{p < r < q\}$ by*

$$\text{var } \mathbb{p} = 2Mf(r) - Mf(p) - Mf(q).$$

III. *We define the variation of a system \mathbb{P} of peaks by*

$$\text{var } \mathbb{P} = \sum_{\mathbb{p} \in \mathbb{P}} \text{var } \mathbb{p}.$$

IV. *We call a peak \mathbb{p} essential if*

$$\max_{p < k < q} f(k) \leq Mf(r) - \frac{1}{4} \text{var } \mathbb{p}.$$

V. *We define averaging operators of radius k for a non-negative integer k by*

$$A_k f(n) = \frac{1}{2k+1} \sum_{j=-k}^k f(n+j)$$

VI. *We define the radius of an essential peak as*

$$\omega(r) := \max\{w > 0 : A_w f(r) = Mf(r)\}$$

Clearly the last part of the definition needs further elaboration. We need to know that the set under consideration is not empty and that it contains finitely many elements. These as well as a further property of $\omega(r)$ shall be dealt with below, but first we introduce some further notation. For $x, y \in \mathbb{Z}$ the notation $[x, y]$ will stand for integers n satisfying $x \leq n \leq y$, and we will call $[x, y]$ an interval. By

length of an interval $[x, y]$ we will mean the quantity $y - x$. We define average of a function f on an interval $[x, y]$ by

$$A_{x,y}f = \frac{1}{y-x+1} \sum_{k=x}^y f(k).$$

Now we state and prove the lemma clarifying the last part of the Definition 1.

Lemma 1. *Let $\mathbb{p} = \{p < r < q\}$ be an essential peak. Then $w(r)$ is well defined and satisfies*

$$r - \omega(r) < p < q < r + \omega(r).$$

Proof. First let's see that our set is nonempty. Since \mathbb{p} is an essential peak we have $f(r) < Mf(r)$. Thus for our set to be empty we must have for every $\omega \geq 0$ $A_{\omega}f(r) < Mf(r)$. But then also by definition of the maximal function we must have a strictly increasing sequence $\{\omega_j\}_{j \in \mathbb{N}}$ of natural numbers such that

$$\lim_{j \rightarrow \infty} A_{\omega_j}f(r) = Mf(r).$$

But note that we also have

$$Mf(p) \geq \lim_{j \rightarrow \infty} A_{\omega_j}f(p) = Mf(r) > Mf(p).$$

Note that this same argument also gives that our set cannot contain infinitely many elements, hence $\omega(r)$ is well defined.

Now note that $f(p) \leq Mf(p) < Mf(r)$ and $f(q) \leq Mf(q) < Mf(r)$, thus $p \leq r - \omega(r) < r + \omega(r) \leq q$ would imply $A_{\omega(r)}f(r) < Mf(r)$, hence at least one of $r - \omega(r) < p$, $q < r + \omega(r)$ is true. We assume the first one is true, and the second is wrong: the converse can be dealt with similarly. We have $A_{p+\omega(r)-r}f(p) \leq Mf(p) < Mf(r)$, which means $A_{2p+\omega(r)-r+1, r+\omega(r)}f \geq Mf(r)$. But $p < 2p+\omega(r)-r+1 < r+\omega(r) \leq q$ means $A_{2p+\omega(r)-r+1, r+\omega(r)}f < Mf(r)$. Hence we are done. \square

The following is the lemma that handles variation arising from a specific class of local extrema. As shall be explained, it plays a fundamental role in our proof.

Lemma 2. *Let $[x, y]$ be an interval of length L with L an even integer. Let $\mathbb{p}_i = \{p_i < r_i < q_i\}$ be a system of essential peaks satisfying*

$$x \leq r_1 < q_1 \leq p_2 < r_2 < q_2 \leq \dots \leq p_{m-1} < r_{m-1} \leq p_m < r_m \leq y$$

and $32L < w(r_i) \leq 64L$ for $1 \leq i \leq m$. Then there exists $s < u < v < t$ such that,

$$x - 64L \leq s, \quad t \leq y + 64L, \quad u - s \geq 4L, \quad v - u = L, \quad t - v \geq 4L$$

$$\min\{f(s), f(t)\} - A_{u,v}f \geq \frac{1}{12} \sum_{i=1}^m \text{var } \mathbb{p}_i$$

This lemma says that if in a system of essential peaks, all peaks lie in an interval of length comparable to all of their radii, the variation of this system can be bounded by using values of the function at nearby points. So this immediately implies that we can put together such systems located at sufficiently distant intervals easily. Hence even if we do not require the peaks to lie in an interval of certain length, the system can be broken into subsystems using a finite covering of the real line by equally spaced intervals and then easily dominated by the variation of the function. As we will see in the section 3, it is very easy to estimate the variation of non-essential

peaks, so proving this lemma reduces the problem to taking care of systems with essential peaks of different length scales.

Proof. We shall decompose the system into three parts. If we take the first and the last peaks out, there remains a system which entirely lies in the interval. Thus proving the lemma for single peaks and systems lying in the interval, with constant on the right hand side $1/4$ instead of $1/12$ suffices. We will first prove the lemma for a single peak so let $\mathfrak{p} = \{p < r < q\}$ denote our system. Our first step is to find s, t satisfying

$$\begin{aligned} f(s) &\geq Mf(r), & x - 64L &\leq s \leq 2q - (r + \omega(r)) \\ f(t) &\geq Mf(r), & 2p - (r - \omega(r)) &\leq t \leq y + 64L \end{aligned}$$

We shall utilize the same ideas as used in Lemma 1 and since the same procedure deals with both, we shall find an s only.

$$A_{\omega(r)}f(r) = Mf(r), \quad A_{r+\omega(r)-q}f(q) \leq Mf(q) < Mf(r)$$

thus

$$A_{r-\omega(r), 2q-(r+\omega(r))-1}f \geq Mf(r)$$

Since $x - 64L \leq r - \omega(r)$, there must be an s with desired properties.

To locate suitable u, v we shall consider two subcases:

I. If $q - p < 12L$ then

$$s < 2q - (r + \omega(r)) < 2p + 24L - r - 32L < p - 8L.$$

Similarly

$$t > 2p - (r - \omega(r)) > 2q - 24L - r + \omega(r) > q + 8L.$$

So we set $u = p - L/2$, $v = p + L/2$ if $Mf(p) \leq Mf(q)$, and $u = q - L/2$, $v = q + L/2$ otherwise. This choice clearly satisfies distance requirements, and

$$\min\{f(s), f(t)\} - A_{u,v}f \geq Mf(r) - \min\{Mf(p), Mf(q)\} \geq \frac{1}{2} \text{var } \mathfrak{p}$$

II. Let $q - p \geq 12L$. Since \mathfrak{p} is an essential peak

$$f(s), f(t) \geq Mf(r) > \max_{p < k < q} f(k).$$

Thus $s \leq p < q \leq t$. Choosing

$$u = \left\lfloor \frac{p+q}{2} \right\rfloor - L/2, \quad v = \left\lfloor \frac{p+q}{2} \right\rfloor + L/2$$

Then

$$\begin{aligned} u - s &\geq \frac{p+q}{2} - 1 - L/2 - p \geq \frac{q-p}{2} - L/2 - 1 \geq 5L - 1 \geq 4L \\ t - v &\geq q - \frac{p+q}{2} - L/2 \geq 5L \end{aligned}$$

and

$$\min\{f(s), f(t)\} - A_{u,v}f \geq Mf(r) - \max_{p < k < q} f(k) \geq \frac{1}{4} \text{var } \mathfrak{p}.$$

Now assume that our peaks are entirely contained in $[x, y]$, so $x \leq p_1, q_m \leq y$. We will work with a modification of our system: set

$$\begin{aligned} e_i &= p_i, & i = 1 \text{ or } Mf(p_i) &\leq Mf(q_{i-1}) \\ e_i &= q_{i-1}, & i = m+1 \text{ or } Mf(p_i) &> Mf(q_{i-1}) \end{aligned}$$

and

$$\tilde{\mathbb{P}}_i = \{e_i < r_i < e_{i+1}\}, \quad 1 \leq i \leq m.$$

We will show the existence of s_i, t_i for $1 \leq i \leq m$ that satisfy

$$\begin{aligned} f(s_i) &\geq Mf(e_{i+1}) + \frac{Mf(r_i) - Mf(e_{i+1})}{e_{i+1} - r_i} \cdot \omega(r_i), \quad x - 64L \leq s_i \leq x - 30L, \\ f(t_i) &\geq Mf(e_i) + \frac{Mf(r_i) - Mf(e_i)}{r_i - e_i} \cdot \omega(r_i), \quad y + 30L \leq t_i \leq y + 64L. \end{aligned}$$

We will find s_i , and t_i are found similarly. We have

$$\begin{aligned} A_{\omega(r_i)} f(r_i) &= (2\omega(r_i) + 1)Mf(r_i), \\ A_{r+\omega(r_i)-e_i} f(e_i) &\leq (2(r_i + \omega(r_i) - e_i) + 1)Mf(e_{i+1}). \end{aligned}$$

So

$$A_{r_i - \omega(r_i), 2e_i - r_i - \omega(r_i) - 1} f \geq (2\omega(r_i) + 1)(Mf(r_i) - Mf(e_{i+1})) + 2(e_i - r_i)Mf(e_{i+1})$$

Since $x - 64L \leq r_i - \omega(r_i)$ and $2e_i - r_i - \omega(r_i) - 1 \leq 2y - x - 32L = x - 30L$ there exists an s_i with asserted properties.

To locate u, v we consider two cases.

I. Let

$$|Mf(e_{m+1}) - Mf(e_1)| > \frac{1}{2} \sum_{i=1}^m \text{var } \tilde{\mathbb{P}}_i.$$

Let's also assume that $Mf(e_{m+1}) > Mf(e_1)$, the other case is similar. Since $Mf(e_1) \geq f(e_1)$, if we can show that $f(s_i), f(t_j) \geq Mf(e_{m+1})$ for some i, j choosing $s = s_i, t = t_j, u = e_1 - L/2, v = e_1 + L/2$ will do. From our choice of s_i we have $Mf(s_m) \geq Mf(e_{m+1})$ and

$$\begin{aligned} f(t_m) &\geq Mf(e_m) + \frac{Mf(r_m) - Mf(e_m)}{r_m - e_m} \cdot (r_m - e_m) = Mf(r_m) \\ &\geq Mf(e_{m+1}) \end{aligned}$$

II. Let

$$|Mf(e_{m+1}) - Mf(e_1)| \leq \frac{1}{2} \sum_{i=1}^m \text{var } \tilde{\mathbb{P}}_i.$$

We know that

$$Mf(e_{m+1}) - Mf(e_1) = \sum_{i=1}^m (Mf(r_i) - Mf(e_i)) - (Mf(r_i) - Mf(e_{i+1}))$$

and that

$$\sum_{i=1}^m \text{var } \tilde{\mathbb{P}}_i = \sum_{i=1}^m (Mf(r_i) - Mf(e_i)) + (Mf(r_i) - Mf(e_{i+1})).$$

Thus we have

$$\sum_{i=1}^m Mf(r_i) - Mf(e_i), \quad \sum_{i=1}^m Mf(r_i) - Mf(e_{i+1}) \geq \frac{1}{4} \sum_{i=1}^m \text{var } \tilde{\mathbb{P}}_i,$$

We choose i_0, j_0 to be the indices that maximize the expressions

$$\frac{Mf(r_i) - Mf(e_{i+1})}{e_{i+1} - r_i}, \quad \frac{Mf(r_j) - Mf(e_j)}{r_j - e_j}.$$

Then we have

$$\begin{aligned}
f(s_{i_0}) - Mf(e_{i_0+1}) &\geq \frac{Mf(r_{i_0}) - Mf(e_{i_0+1})}{e_{i_0+1} - r_{i_0}} \cdot \omega(r_{i_0}) \\
&\geq \frac{Mf(r_{i_0}) - Mf(e_{i_0+1})}{e_{i_0+1} - r_{i_0}} \cdot 32L \\
&\geq \frac{Mf(r_{i_0}) - Mf(e_{i_0+1})}{e_{i_0+1} - r_{i_0}} \cdot 32 \cdot \sum_{i=1}^m e_{i+1} - r_i \\
&= 32 \sum_{i=1}^m \frac{Mf(r_i) - Mf(e_{i+1})}{e_{i+1} - r_i} \cdot (e_{i+1} - r_i) \\
&\geq 32 \sum_{i=1}^m Mf(r_i) - Mf(e_{i+1}) \\
&\geq 8 \sum_{i=1}^m \text{var } \tilde{\mathbb{P}}_i.
\end{aligned}$$

The same process applies to $f(t_{j_0}) - Mf(e_{j_0})$. So set $s = s_{i_0}, t = t_{j_0}, u = e_{j_0} - L/2, v = e_{j_0} + L/2$ and note that

$$|Mf(e_{j_0}) - Mf(e_{i_0+1})| \leq \sum_{i=1}^m \text{var } \tilde{\mathbb{P}}_i.$$

Hence this choice satisfies desired properties. \square

3. BOUNDING SYSTEMS CONTAINING DIFFERENT SCALES

We first fix a system

$$a_1 < b_1 < a_2 < b_2 < \dots < a_\sigma < b_\sigma < a_{\sigma+1}$$

satisfying $Mf(a_i) < Mf(b_i)$ and $Mf(a_{i+1}) < Mf(b_i)$ for $1 \leq i \leq \sigma$. We will use \mathbb{P} to denote collection of all peaks $\mathbb{P}_i = \{a_i < b_i < a_{i+1}\}$ arising from this system. The letter \mathbb{E} will stand for those \mathbb{P}_i that are essential. We further partition the essential peaks as follows: for $n > 5$, $k \in \mathbb{Z}$ we define

$$\mathbb{E}_k^n = \{\mathbb{P}_i \in \mathbb{E} : 2^{n-1} < \omega(b_i) \leq 2^n, k2^{n-5} < b_i \leq (k+1)2^{n-5}\},$$

and we let \mathbb{E}' denote all essential peaks not belonging to one of the above collections.

We first will bound the variation of non-essential peaks, and then describe how to handle \mathbb{E}' . After these two relatively easy tasks we will set ourselves to bounding the variation of remaining peaks.

Lemma 3. *We have the inequality*

$$\text{var}(\mathbb{P} \setminus \mathbb{E}) \leq 2 \text{var } f.$$

Proof. Since $\mathbb{P} \in \mathbb{P} \setminus \mathbb{E}$ is a non-essential peak we have a point $x_i \in [a_i + 1, a_{i+1} - 1]$ satisfying

$$f(x_i) \geq Mf(b_i) - \frac{1}{4} \text{var } \mathbb{P}_i.$$

Then we have

$$\begin{aligned}
|f(x_i) - f(a_i)| + |f(x_i) - f(a_{i+1})| &\geq 2f(x_i) - f(a_i) - f(a_{i+1}) \\
&\geq 2(Mf(b_i) - \frac{1}{4} \text{var } \mathbb{p}_i) - Mf(a_i) - Mf(a_{i+1}) \\
&= \frac{1}{2} \text{var } \mathbb{p}_i.
\end{aligned}$$

From this our assertion is clear. \square

Lemma 4. *We have*

$$\text{var } \mathbb{E}' \leq 1200 \cdot \text{var } f.$$

Proof. We partition the integers into subsets $\mathbb{Z}_l = 300\mathbb{Z} + l$ for $0 \leq l < 300$. Similarly partition \mathbb{E}' into

$$\mathbb{E}'_l = \{\mathbb{p}_i = \{a_i < b_i < a_{i+1}\} : \mathbb{p}_i \in \mathbb{E}', b_i \in \mathbb{Z}_l\}.$$

We apply to \mathbb{p}_i the same procedure as in the proof of Lemma 2. for a single peak to find $s_i < u_i < t_i$ satisfying $b_i - 32 \leq s_i, t_i \leq b_i + 32$ and

$$\min\{f(s_i), f(t_i)\} - f(u_i) \geq \frac{1}{4} \text{var } \mathbb{p}_i.$$

Using these points we have

$$\text{var } \mathbb{E}'_l = \sum_{\mathbb{p}_i \in \mathbb{E}'_l} \text{var } \mathbb{p}_i \leq 4 \cdot \sum_{\{i: \mathbb{p}_i \in \mathbb{E}'_l\}} |f(s_i) - f(u_i)| + |f(t_i) - f(u_i)| \leq 4 \cdot \text{var } f$$

since the peaks in \mathbb{E}'_l are sufficiently distant. Thus

$$\text{var } \mathbb{E}' = \sum_l \text{var } \mathbb{E}'_l \leq 300 \cdot 4 \cdot \text{var } f = 1200 \cdot \text{var } f.$$

\square

To handle the remaining peaks we need to classify further. The next lemma will serve to this purpose.

Lemma 5. *Let \mathbb{E}_k^n be non-empty for some $n \geq 6, k \in \mathbb{Z}$. Then one of the following is true:*

A. *There exists $s < \alpha < \beta < \gamma < \delta < t$ satisfying*

$$\begin{aligned}
(k-64)2^{n-5} &\leq s, \quad t \leq (k+65)2^{n-5}, \\
\alpha - s &\geq 2^{n-5}, \quad \beta - \alpha \geq 2^{n-5}, \quad \gamma - \beta \geq 2^{n-4}, \quad \delta - \gamma \geq 2^{n-5}, \quad t - \delta \geq 2^{n-5}
\end{aligned}$$

$$\min\{f(s), f(t)\} - \max\{A_{\alpha, \beta} f, A_{\gamma, \delta} f\} \geq \frac{1}{24} \text{var } \mathbb{E}_k^n$$

B. *There exists $\alpha < \beta < u < v < \gamma < \delta$ satisfying*

$$\begin{aligned}
(k-64)2^n &\leq \alpha, \quad \delta \leq (k+65)2^n, \\
\beta - \alpha &\geq 2^{n-5}, \quad u - \beta \geq 2^{n-5}, \quad v - u \geq 2^{n-5}, \quad \gamma - v \geq 2^{n-5}, \quad \delta - \gamma \geq 2^{n-5}
\end{aligned}$$

$$\min\{A_{\alpha, \beta} f, A_{\gamma, \delta} f\} - A_{u, v} f \geq \frac{1}{24} \text{var } \mathbb{E}_k^n.$$

Proof. We have by Lemma 2 points $s < u < v < t$ for peaks of \mathbb{E}_k^n and interval $[k2^{n-5}, (k+1)2^{n-5}]$. We then define

$$\alpha = u - 3 \cdot 2^{n-5}, \quad \beta = u - 2 \cdot 2^{n-5}, \quad \gamma = v + 2 \cdot 2^{n-5}, \quad \delta = 3 \cdot 2^{n-5}.$$

If the inequality

$$\min\{A_{\alpha,\beta}f, A_{\gamma,\delta}f\} \geq \frac{1}{2} \min\{f(s), f(t)\} + \frac{1}{2} A_{u,v}f$$

is satisfied then we just need to subtract $A_{u,v}f$ from both sides and use the Lemma 2 to see **B** satisfied. Assume it does not hold. We first assume $A_{\alpha,\beta} = \min\{A_{\alpha,\beta}f, A_{\gamma,\delta}f\}$. In this case

$$\max\{A_{\alpha,\beta}, A_{u,v}\} + \frac{1}{2} \min\{f(s), f(t)\} \leq \min\{f(s), f(t)\} + \frac{1}{2} A_{u,v}f$$

Applying Lemma 2 from here yields the desired inequality if we keep α, β the same, and set $\gamma = u, \delta = v$. For the case $A_{\gamma,\delta} = \min\{A_{\alpha,\beta}f, A_{\gamma,\delta}f\}$ all we need is to keep γ, δ the same and set $\alpha = u, \beta = v$. \square

Thus we define \mathcal{A} to be the union of \mathbb{E}_k^n satisfying **A**, and \mathcal{B} as the union those satisfying **B**. We further define \mathcal{A}_K^n to be the union of \mathbb{E}_k^n in \mathcal{A} for which $k \equiv \text{mod } 300$, and \mathcal{B}_K^n is defined analogously. Notice that since we took a finite number of peaks in , there exists n_A representing the largest n for which \mathcal{A}_K^n is non-empty for at least one K . Similarly we have an n_B . In the next two sections we shall deal respectively with variation arising from peaks of \mathcal{A} and \mathcal{B} .

4. THE VARIATION OF PEAKS OF \mathcal{A}

The following is the main proposition we want to prove in this section.

Proposition 1. *Let $0 \leq N \leq 11$, $0 \leq K \leq 299$ and let L_N denote 2^{N-6} if $N \geq 6$ and 2^{N+6} if $n \leq 5$. There exists a system*

$$x_1 < u_1 < v_1 < x_2 < u_2 < v_2 < x_3 < \dots < x_m < u_m < v_m < x_{m+1}$$

with properties

$$u_i - x_i \geq L_N, \quad v_i - u_i \geq L_N, \quad x_{i+1} - v_i \geq L_N, \quad 1 \leq i \leq m,$$

$$\sum_{i=1}^m f(x_i) + f(x_{i+1}) - 2A_{u_i, v_i}f \geq \frac{1}{60} \sum_{n=N \pmod{12}} \text{var } \mathcal{A}_K^n.$$

We shall prove this inductively. Let $n_N = N \pmod{12}$ denote the maximum integer n for which \mathcal{A}_K^n is non-empty. We clearly have a system as described above that bounds the variation of \mathcal{A}_K^n which have 2^{n_N-5} instead of L_N -this is true only if $n_N > N$ of course, but if $n_N = N$ we directly obtain the desired system using Lemma 2-. Now assume we have a system that bounds sum of variations coming from all classes \mathcal{A}_K^n for $n > n_0$ where $\mathcal{A}_K^{n_0}$ is non-empty, and that has 2^{n_0+12} instead of L_N . If we can modify this system so that it bounds all classes for $n \geq n_0$ with 2^{n_0} replacing L_N , a finite iteration would give our proposition.

Thus we assume there exists a system

$$(3) \quad x_1 < u_1 < v_1 < x_2 < u_2 < v_2 < x_3 < \dots < x_{m_0} < u_{m_0} < v_{m_0} < x_{m_0+1}$$

that satisfies conditions given by the inductive hypothesis above. The class $\mathcal{A}_K^{n_0}$ is a union of a finite number of systems of peaks $\mathbb{E}_k^{n_0}$, we will describe how to incorporate these into the existing system. Pick one such $\mathbb{E}_k^{n_0}$ and consider $s <$

$\alpha < \beta < \gamma < \delta < t$ coming from the alternative **A** of Lemma 5 for it. We will modify (3) according to its relation with the interval $[s, t]$.

I. First assume for any $1 \leq i \leq m_0$ we have $\text{dist}([u_i, v_i], [s, t]) \geq 2^{n-5}$. In this case one of the intervals

$$(-\infty, u_1], [v_1, u_2], \dots [v_{m_0-1}, u_{m_0}], [v_0, \infty)$$

must contain $[s, t]$. This interval also contain a unique $x_i, 1 \leq i \leq m_0$, which must satisfy either $\text{dist}(x_i, \beta) \geq \text{dist}(x_i, \gamma)$ or $\text{dist}(x_i, \beta) < \text{dist}(x_i, \gamma)$. If the first happens we take s, α, β , otherwise we take γ, δ, t and add them to our system. The new system is easily seen to satisfy desired properties.

II. There exists an i with $\text{dist}([u_i, v_i], [s, t]) < 2^{n-5}$. We first note that this i is unique. Observe that either $(k - 150)2^{n-5} \in [u_i, v_i]$ or $(k + 150)2^{n-5} \in [u_i, v_i]$, we will assume the first, as the second is handled similarly. Let $g = h = k \pmod{300}$ be such that $(g - 155)2^{n-5} \leq x' < (g + 145)2^{n-5}, (h - 155)2^{n-5} \leq y' < (h + 145)2^{n-5}$. Notice that these condition determine g, h uniquely. Using these we partition our interval

$$[u_i, (g + 150)2^{n-5} - 1], [(g + 150)2^{n-5}, (g + 450)2^{n-5} - 1], \dots [(h - 150)2^{n-5}, v_i].$$

One of these subintervals contains $(k - 150)2^{n-5}$ which will be denoted by I and, average of f on one of these subintervals is less than or equal to average over $[u_i, v_i]$, we will call this $[u'_i, v'_i]$. If I is not the same as $[u'_i, v'_i]$, then this latter interval is distant enough from $[s, t]$, and replacing $[u_i, v_i]$ by $[u'_i, v'_i]$ and choosing appropriate ones out of $\{s, \alpha, \beta, \gamma, \delta, t\}$ will do. If they are the same then we have to consider two different cases. Either there exists $[c, d] \subset [u'_i, v'_i]$ with $d - c \geq 2^{n-5}$ such that

$$A_{c,d}f \leq A_{u_i, v_i}f - \frac{1}{120} \text{var } \mathbb{E}_k^{n_0},$$

or we have a system $c < d < y < c' < d'$ with $[c, d'] \subset [u'_i, u'_i + 300 \cdot 2^{n-5}]$ and

$$d - c \geq 2^{n-5}, \quad y - d \geq 2^{n-5}, \quad c' - y \geq 2^{n-5}, \quad d' - c' \geq 2^{n-5},$$

such that

$$A_{c,d}f + A_{c',d'}f - f(y) \leq A_{u_i, v_i}f - \frac{1}{120} \text{var } \mathbb{E}_k^{n_0}.$$

In both cases what to do is clear, in the first case $[u_i, v_i]$ is replaced by $[c, d]$, while in the second we replace $[u_i, v_i]$ by two intervals $[c, d], [c', d']$ and the point y between them. But that one of these must hold should be shown. We set

$$w'_i = u'_i + \left\lceil \frac{v'_i - u'_i}{5} \right\rceil$$

and observe that both $[u'_i, w'_i], [w'_i + 1, v'_i]$ are longer than 2^{n-5} . We have either

$$(4) \quad A_{w'_i+1, v'_i}f \leq A_{u'_i, v'_i}f - \frac{1}{120} \text{var } \mathbb{E}_k^{n_0} \quad \text{or} \quad A_{u'_i, w'_i}f \leq A_{u'_i, v'_i}f + \frac{4}{120} \text{var } \mathbb{E}_k^{n_0},$$

and if the first holds we just set $c = w'_i + 1, d = v'_i$ to obtain **a** whereas if the second holds we let $c = u'_i, d = w'_i, y = s, c' = \alpha, d' = \beta$. That $y - d \geq 2^{n-5}$ follows from the definitions of u'_i, w'_i .

We thus incorporated the first $\mathbb{E}_k^{n_0}$ into the system. For the rest we apply a similar procedure but, we also have to deal with previously made changes, which shorten the distance between successive points from 2^{n+7} to 2^{n-5} . Let us incorporate a second system $\mathbb{E}_l^{n_0}$. Let our modified system be

$$(5) \quad x_1 < u_1 < v_1 < x_2 < u_2 < v_2 < x_3 < \dots < x_{m_0,1} < u_{m_0,1} < v_{m_0,1} < x_{m_0,1+1}$$

and consider $s' < \alpha' < \beta' < \gamma' < \delta' < t'$ coming from Lemma 5 for $\mathbb{E}_l^{n_0}$. We again have the same two alternatives which this time we will call **I'**, **II'**, and if **I'** is the case, exactly same ideas suffice. If on the other hand $\text{dist}([u_i, v_i], [s', t']) \leq 2^{n-5}$ holds for some $1 \leq i \leq m_{0,1}$, then some additional consideration is needed. First we need to see that this $[u_i, v_i]$ is unique. Since $k \neq l$ we have $\text{dist}([s', t'], [(k-150)2^{n-5}(k+150)2^{n-5}]) \geq 2^{n-5}$ such $[u_i, v_i]$ can be either unmodified intervals, or only first of three types of intervals arising from **II**. If $[u_i, v_i]$ is close to an unmodified interval, it is sufficiently distant from all other unmodified intervals and intervals arising from **II**. Similarly being close to an interval arising from **II** guarantees distance from all unmodified intervals. Thus $[u_i, v_i]$ is unique. After this methods described in **II** handles both cases. Clearly these considerations suffice to add the remaining systems, and after a finite number of steps we will have $\mathcal{A}_K^{n_0}$ incorporated.

Thus the proof of our proposition is complete. From this proposition we easily deduce that

$$\text{var } \mathcal{A} \leq 120 \cdot 2^{12} \cdot 300 \cdot \text{var } f.$$

5. THE VARIATION OF PEAKS OF \mathcal{B}

Arguments of this section will largely be analogous to those of section 4. We state the main proposition of this section.

Proposition 2. *Let $0 \leq N \leq 11$, $0 \leq K \leq 299$ and let L_N denote 2^{N-6} if $N \geq 6$ and 2^{N+6} if $n \leq 5$. There exists a system*

$$x_1 < y_1 < u_1 < v_1 < x_2 < y_2 < u_2 < v_2 < \dots < u_m < v_m < x_{m+1} < y_{m+1}$$

with properties

$$\begin{aligned} y_i - x_i &\geq L_N, \quad 1 \leq i \leq m+1, \\ u_i - y_i &\geq L_N, \quad v_i - u_i \geq L_N, \quad x_{i+1} - v_i \geq L_N \quad 1 \leq i \leq m, \\ \sum_{i=1}^m A_{x_i, y_i} f + A_{x_{i+1}, y_{i+1}} f - 2A_{u_i, v_i} f &\geq \frac{1}{60} \sum_{n=N \bmod 12} \text{var } \mathcal{B}_K^n. \end{aligned}$$

We shall again utilize induction. Assume we have a system

$$x_1 < y_1 < u_1 < v_1 < x_2 < y_2 < u_2 < v_2 < \dots < u_{m_0} < v_{m_0} < x_{m_0+1} < y_{m_0+1}$$

that bounds the variation of all classes \mathcal{B}_K^n for $n > n_0$ where $\mathcal{B}_K^{n_0}$ is non-empty, and that has 2^{n_0+12} instead of L_N . Let $\mathbb{E}_k^{n_0}$ be one of subsystems comprising $\mathcal{B}_K^{n_0}$, and consider $\alpha < \beta < u < v < \gamma < \delta$ coming from the alternative **B** of Lemma 5 for it. We again will investigate the relation of our system with the interval $[\alpha, \delta]$, this time however, we will have three cases.

I. First assume for all $1 \leq i \leq m_0$ we have $\text{dist}([u_i, v_i], [\alpha, \delta]) \geq 2^{n-5}$, and for all $1 \leq i \leq m_0+1$ we have $\text{dist}([x_i, y_i], [\alpha, \delta]) \geq 2^{n-5}$. This case is easy, we just choose two appropriate ones out of three intervals $[\alpha, \beta]$, $[u, v]$, $[\gamma, \delta]$, and incorporate to our system.

II. There exist an $1 \leq i \leq m_0$ such that $\text{dist}([u_i, v_i], [\alpha, \delta]) < 2^{n-5}$. Clearly this i is unique, moreover $[\alpha, \delta]$ is distant from $[x_i, y_i]$ type intervals. This case will be dealt with in the same way as the case **II** of section 4. We divide $[u_i, v_i]$ into subintervals and pick I , $[u'_i, v'_i]$ exactly in the same way. The case when they are not the same is easy and handled as before, whereas if they are the same either there exists $[c, d] \subset [u'_i, v'_i]$ with $d - c \geq 2^{n-5}$ such that

$$(6) \quad A_{c,d}f \leq A_{u_i,v_i}f - \frac{1}{120} \text{var } \mathbb{E}_k^{n_0},$$

or we have a system $c < d < x < y < c' < d'$ with $[c, d'] \subset [u'_i, u'_i + 300 \cdot 2^{n-5}]$ and $d - c \geq 2^{n-5}$, $x - d \geq 2^{n-5}$, $y - x \geq 2^{n-5}$, $c' - y \geq 2^{n-5}$, $d' - c' \geq 2^{n-5}$, such that

$$(7) \quad A_{c,d}f + A_{c',d'}f - A_{x,y}f \leq A_{u_i,v_i}f - \frac{1}{120} \text{var } \mathbb{E}_k^{n_0}.$$

In each what to do is clear, we will show that one of these holds. Defining w'_i as before we have the dichotomy given in (4). If the first alternative of this dichotomy holds we set $c = w'_i + 1, d = v'_i$ and get (6), while if the second holds we set

$$(8) \quad c = u'_i, \quad d = w'_i, \quad x = u, \quad y = v, \quad c' = \gamma, \quad d' = \delta$$

and obtain (7).

III. There exist an $1 \leq i \leq m_0 + 1$ such that $\text{dist}([x_i, y_i], [\alpha, \delta]) < 2^{n-5}$. This case is similar to what we have above, only essential difference will be changes in signs of averages over intervals. As above this i is unique, further $[\alpha, \delta]$ is distant from $[u_i, v_i]$ type intervals. We subdivide $[x_i, y_i]$ the way we did $[u_i, v_i]$ above and, choose I . This time, however, $[x'_i, y'_i]$ will be the subinterval on which average is not smaller than the average over $[x_i, y_i]$. If these are not the same, replacing $[x_i, y_i]$ with $[x'_i, y'_i]$ will suffice. If they are the same we either have $[c, d] \subset [x'_i, y'_i]$ with $d - c \geq 2^{n-5}$ such that

$$A_{c,d}f \geq A_{x_i,y_i}f + \frac{1}{120} \text{var } \mathbb{E}_k^{n_0},$$

or we have a system $c < d < \mu < \nu < c' < d'$ with $[c, d'] \subset [x'_i, x'_i + 300 \cdot 2^{n-5}]$ and $d - c \geq 2^{n-5}$, $\mu - d \geq 2^{n-5}$, $\nu - \mu \geq 2^{n-5}$, $c' - \nu \geq 2^{n-5}$, $d' - c' \geq 2^{n-5}$, such that

$$A_{c,d}f + A_{c',d'}f - A_{\mu,\nu}f \geq A_{x_i,y_i}f + \frac{1}{120} \text{var } \mathbb{E}_k^{n_0},$$

$$A_{c,d}f, A_{c',d'}f \geq A_{\mu,\nu}f$$

This last additional property handles problems arising when $i = 1$ and $i = m_0 + 1$. As before in the first case $[c, d]$ replaces $[x_i, y_i]$, while in the second $[c, d], [\mu, \nu], [c', d']$, does. Defining

$$z'_i = x'_i + \left\lceil \frac{y'_i - x'_i}{5} \right\rceil$$

we have the dichotomy

$$A_{z'_i+1,y'_i}f \geq A_{x'_i,y'_i}f + \frac{1}{120} \text{var } \mathbb{E}_k^{n_0} \quad \text{or} \quad A_{u'_i,z'_i}f \geq A_{x'_i,y'_i}f - \frac{4}{120} \text{var } \mathbb{E}_k^{n_0}.$$

If the first is the case we just set $c = z'_i + 1, d = y'_i$ to obtain (6), if the second holds we set $\mu = u, \quad \nu = v, \quad c' = \gamma, \quad d' = \delta$, and

$$c = \alpha, \quad d = \beta \quad \text{if} \quad A_{\alpha,\beta}f \geq A_{x'_i,y'_i}f,$$

$$c = x'_i, \quad d = y'_i \quad \text{if} \quad A_{x'_i,y'_i}f > A_{\alpha,\beta}f.$$

Here using the interval on which average is greater guarantees the last additional property.

We thus incorporated $\mathbb{E}_k^{n_0}$ into our system. To incorporate the rest we have to deal with previously made changes. Let us incorporate a second system $\mathbb{E}_l^{n_0}$. Let our modified system be

$$(9) \quad x_1 < y_1 < u_1 < v_1 < x_2 < u_2 < \dots < u_{m_0,1} < v_{m_0,1} < x_{m_0,1+1} < y_{m_0,1+1}$$

and $\alpha' < \beta' < u' < v' < \gamma' < \delta'$ coming from Lemma 5 for $\mathbb{E}_l^{n_0}$. We have the same three alternatives which we will call **I'**, **II'**, **III'**, and if **I'** is the case, exactly same ideas suffice. If **II'** is the case, that is if $[\alpha, \delta]$ is close to $[u_i, v_i]$ for some $1 \leq i \leq m_0$, then this is either an unmodified interval, or emerges as the first of three types of intervals arising from **II**. In either case i should be unique by the same considerations as in section 4, and methods explained in **II** handles this case. If **III'** holds then by the same arguments $[\alpha, \delta]$ is close to $[x_i, y_i]$ for a unique $1 \leq i \leq m_0 + 1$, and this $[x_i, y_i]$ is either unmodified, or a result of a modification through first of three methods described in **III**. In either case methods of **III** deals with this case. Clearly these considerations suffice to add the remaining systems, and after a finite number of steps we will have $\mathcal{B}_K^{n_0}$ added.

This completes the proof of our proposition from which we easily obtain

$$\text{var } \mathcal{B} \leq 120 \cdot 2^{12} \cdot 300 \cdot \text{var } f.$$

6. PROOF OF THEOREM 1

We now use results we proved in sections 3,4,5 to prove our theorem. We have

$$\text{var } Mf \leq \sum_{k=-\infty}^{\infty} |f(k+1) - f(k)| \leq \sup_{m,n: m \leq n} \sum_{k=m}^n |f(k+1) - f(k)|.$$

For each couple m, n with $m \leq n$ dispensing with redundant elements the interval $[m, n+1]$ gives a system

$$b_0 \leq a_1 < b_1 < a_2 < b_2 < \dots < a_\sigma < b_\sigma < a_{\sigma+1} \leq b_{\sigma+1}$$

with $Mf(a_i) < Mf(b_i)$, $Mf(a_{i+1}) < Mf(b_i)$ for $1 \leq i \leq \sigma$, and $Mf(a_1) \leq Mf(b_0)$, $Mf(a_{\sigma+1}) \leq Mf(b_{\sigma+1})$

$$\begin{aligned} \sum_{k=m}^n |f(k+1) - f(k)| &= \sum_{i=1}^{\sigma} (2Mf(b_i) - Mf(a_{i+1}) - Mf(a_i)) \\ &\quad + Mf(b_0) - Mf(a_1) + Mf(a_{\sigma+1}) - Mf(b_{\sigma+1}) \end{aligned}$$

We apply Lemma 3, Proposition 1, Proposition 2 to obtain

$$\sum_{i=1}^{\sigma} (2Mf(b_i) - Mf(a_{i+1}) - Mf(a_i)) \leq (2 \cdot 120 \cdot 2^{12} \cdot 300 + 2) \cdot \text{var } f.$$

On the other hand

$$\begin{aligned} Mf(b_0) - Mf(a_1) + Mf(a_{\sigma+1}) - Mf(b_{\sigma+1}) &\leq 2 \sup_{k \in \mathbb{Z}} Mf(k) - 2 \inf_{k \in \mathbb{Z}} Mf(k) \\ &\leq 2 \sup_{k \in \mathbb{Z}} f(k) - 2 \inf_{k \in \mathbb{Z}} f(k) \\ &\leq 2 \text{var } f \end{aligned}$$

So finally taking supremum on the left we have

$$\text{var } Mf \leq (2 \cdot 120 \cdot 2^{12} \cdot 300 + 4) \cdot \text{var } f.$$

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN URBANA, IL 61820

E-mail address: `temur1@illinois.edu`